

## ELLIPTIC SURFACES WITHOUT 1-HANDLES

KOUICHI YASUI

ABSTRACT. Harer-Kas-Kirby conjectured that every handle decomposition of the elliptic surface  $E(1)_{2,3}$  requires both 1- and 3-handles. We prove that the elliptic surface  $E(n)_{p,q}$  has a handle decomposition without 1-handles for  $n \geq 1$  and  $(p, q) = (2, 3), (2, 5), (3, 4), (4, 5)$ .

## 1. INTRODUCTION

It is not known whether or not the 4-sphere  $S^4$  and the complex projective plane  $\mathbf{CP}^2$  admit an exotic smooth structure. If such a structure exists, then each handle decomposition of it has at least either a 1- or 3-handle (cf. [8]). On the contrary, many simply connected closed topological 4-manifolds are known to admit infinitely many different smooth structures which have neither 1- nor 3-handles in their handle decompositions (cf. Gompf-Stipsicz [4]).

Problem 4.18 in Kirby's problem list [6] is the following: "Does every simply connected, closed 4-manifold have a handlebody decomposition without 1-handles? Without 1- and 3-handles?" It is not known whether or not the simply connected elliptic surface  $E(n)_{p,q}$  ( $n \geq 1$ ,  $p, q \geq 2$ ,  $\gcd(p, q) = 1$ ) admits a handle decomposition without 1-handles. In particular, Harer, Kas and Kirby conjectured in [5] that every handle decomposition of  $E(1)_{2,3}$  requires both 1- and 3-handles. Gompf [3] notes the following: it is a good conjecture that  $E(n)_{p,q}$  ( $p, q \geq 2$ ) has no handle decomposition without 1- and 3-handles.

In [7] and [8] we constructed a homotopy  $E(1)_{2,3}$  which has the same Seiberg-Witten invariant as  $E(1)_{2,3}$  and has a handle decomposition without 1- and 3-handles. Recently Akbulut [1] proved that  $E(1)_{2,3}$  has a handle decomposition without 1- and 3-handles, by using knot surgery on  $E(1)$  and investigating a dual handle decomposition. He also proved that infinitely many different smooth structures on  $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}^2}$  admit handle decompositions without 1-handles.

In this paper, we prove the theorem below by improving our previous procedure ([7], [8]). Our method is different from Akbulut.

**Theorem 1.1.** *The elliptic surface  $E(n)_{p,q}$  has a handle decomposition without 1-handles, for  $n \geq 1$  and  $(p, q) = (2, 3), (2, 5), (3, 4), (4, 5)$ .*

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## 2. RATIONAL BLOW-DOWN

In this section we review the rational blow-down introduced by Fintushel-Stern [2]. See also Gompf-Stipsicz [4].

Let  $C_p$  and  $B_p$  ( $p \geq 2$ ) be the smooth 4-manifolds defined by Kirby diagrams in Figure 1. The boundary  $\partial C_p$  of  $C_p$  is diffeomorphic to the lens space  $L(p^2, p-1)$  and to the boundary  $\partial B_p$  of  $B_p$ .

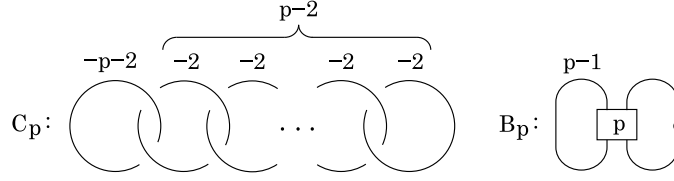


FIGURE 1.

Suppose that  $C_p$  embeds in a smooth 4-manifold  $X$ . Let  $X_{(p)}$  be a smooth 4-manifold obtained from  $X$  by removing  $C_p$  and gluing  $B_p$ . The 4-manifold  $X_{(p)}$  is called the rational blow-down of  $X$  along  $C_p$ . Note that  $X_{(p)}$  is uniquely determined up to diffeomorphism by a fixed pair  $(X, C_p)$ . This operation has the following relation with the logarithmic transformation.

**Theorem 2.1** (Fintushel-Stern [2], see also Gompf-Stipsicz [4]). *Suppose that a smooth 4-manifold  $X$  contains a cusp neighborhood, that is, a 0-handle with a 2-handle attached along a 0-framed right trefoil knot. Let  $X_p$  be the smooth 4-manifold obtained from  $X$  by performing a logarithmic transformation of multiplicity  $p$  in the cusp neighborhood. Then there exists a copy of  $C_p$  in  $X \# (p-1)\overline{\mathbf{CP}^2}$  such that the rational blow-down of  $X \# (p-1)\overline{\mathbf{CP}^2}$  along the copy of  $C_p$  is diffeomorphic to  $X_p$ .*

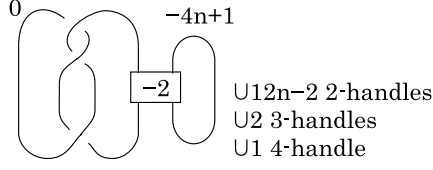
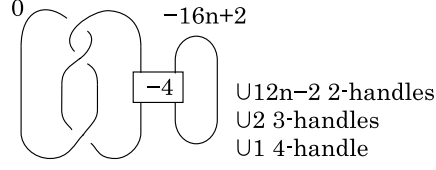
## 3. PROOF

In this section we prove Theorem 1.1. We do not draw (whole) Kirby diagrams of elliptic surfaces. However, one can draw whole diagrams of elliptic surfaces without 1-handles, following the procedures in this section.

Let  $E(n)$  be the simply connected elliptic surface with Euler characteristic  $12n$  and with no multiple fibers, and  $E(n)_{p_1, \dots, p_k}$  the elliptic surface obtained from  $E(n)$  by performing logarithmic transformations of multiplicities  $p_1, \dots, p_k$ .

**Proposition 3.1.** *For  $n \geq 1$ , the elliptic surface  $E(n)_2$  has handle decompositions as in Figure 2 and 3. Each obvious cusp neighborhood in Figure 2 and 3 is isotopic to a regular neighborhood of a cusp fiber of  $E(n)_2$ .*

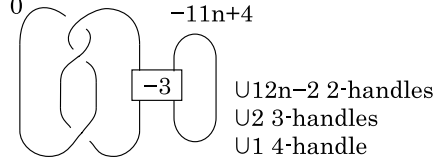
*Proof.*  $E(n)_p$  admits a handle decomposition in Figure 9 (see Gompf-Stipsicz [4, page 315 ~ 316] and Harer-Kas-Kirby [5]). The obvious cusp neighborhood in Figure 9 is isotopic to a regular neighborhood of a cusp fiber of  $E(n)_p$  (see [4] and [5]). Figure 10 is the  $p = 2$  case of Figure 9. Note that we do not draw  $6n - 1$  2-handles in Figure 10. We change Figure 10 into Figure 2 and 3 without sliding the cusp neighborhood over any handles, as follows. In Figure 10, we slide  $-4n + 2$


 FIGURE 2.  $E(n)_2$ 

 FIGURE 3.  $E(n)_2$ 

framed knot over vertical  $-1$  framed knots as shown in Figure 10 ~ 13. Note that  $\frac{k}{2}$  in the boxes denotes  $k$  right half-twists. By repeating handle slides similar to Figure 11 ~ 13, we obtain Figure 14. An isotopy gives Figure 15. By canceling 1-handles, we get Figure 2.

In Figure 15, we slide a vertical  $-1$  framed knot over  $-4n + 1$  framed knot. We get Figure 16. We slide  $-4n$  framed knot over  $-4n + 1$  framed knot as shown in Figure 17. Sliding  $-16n + 3$  framed knot over a vertical  $-1$  framed knot gives Figure 18. By repeating handle slides similar to Figure 11 ~ 13, we obtain Figure 19. An isotopy gives Figure 20. By canceling 1-handles, we get Figure 3.  $\square$

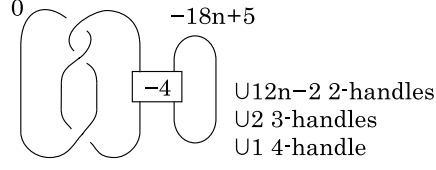
**Proposition 3.2.** *For  $n \geq 1$ , the elliptic surface  $E(n)_3$  admits a handle decomposition as in Figure 4. The obvious cusp neighborhood in Figure 4 is isotopic to a regular neighborhood of a cusp fiber of  $E(n)_3$ .*


 FIGURE 4.  $E(n)_3$ 

*Proof.* Figure 21 is the  $p = 3$  case of Figure 9. We change Figure 21 into Figure 4 without sliding the cusp neighborhood over any handles. In Figure 21, we slide  $-9n + 3$  framed knot over vertical  $-1$  framed knots as shown in Figure 21 ~ 28. An isotopy gives Figure 29. We get Figure 30 by sliding  $-9n + 2$  framed knot over a vertical  $-1$  framed knot. In the  $n \geq 2$  case, we obtain Figure 31 by repeating handle slides similar to Figure 21 ~ 30. We get the  $n = 1$  case of Figure 31 by an isotopy in the  $n = 1$  case of Figure 21. Handle slides similar to Figure 21 ~ 24 give Figure 32. An isotopy gives Figure 33. By canceling 1-handles, we get Figure 4.  $\square$

**Proposition 3.3.** *For  $n \geq 1$ , the elliptic surface  $E(n)_4$  admits a handle decomposition as in Figure 5. The obvious cusp neighborhood in Figure 5 is isotopic to a regular neighborhood of a cusp fiber of  $E(n)_4$ .*

*Proof.* In Figure 9 of  $E(n)_p$ , we repeat handle slides shown in Figure 34. We then get the diagram of  $E(n)_p$  in Figure 35. (This diagram is a key of our proof for  $n \geq 2$ . The way to construct this diagram is suggested by the referee.) Note that we did not slide the cusp neighborhood in Figure 9 over any handles. Figure 36 is

FIGURE 5.  $E(n)_4$ 

the  $p = 4$  case of Figure 35. We change Figure 36 into Figure 5 without sliding the cusp neighborhood over any handles, as follows.

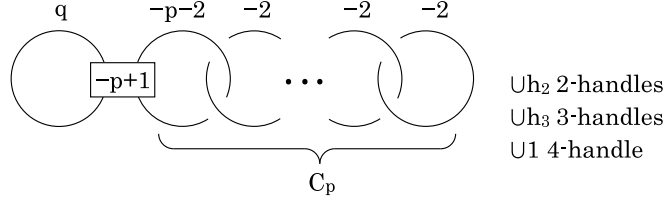
The  $n \geq 2$  case. We slide handles as shown in Figure 36 ~ 46. We then get Figure 47. Isotopies give Figure 48 and 49. We have Figure 53 by handle slide as shown in Figure 49 ~ 52. By repeating handle slides similar to Figure 36 ~ 53, we obtain Figure 54. We slide handles similarly to Figure 36 ~ 43. We then get Figure 55. An isotopy gives Figure 56. By cancelling 1-handles, we have Figure 5.

The  $n = 1$  case. Figure 54 is isotopic to the  $n = 1$  case of Figure 36. We slide handles similarly to Figure 36 ~ 43. We then get Figure 55. An isotopy gives Figure 56. By cancelling 1-handles, we have Figure 5.  $\square$

**Lemma 3.4.** *Suppose that a simply connected closed smooth 4-manifold  $X$  has a handle decomposition as in Figure 6. Here  $q$  is an arbitrary integer.  $h_2$  and  $h_3$  are arbitrary non-negative integers. Let  $X_{(p)}$  be the rational blow-down of  $X$  along the copy of  $C_p$  in Figure 6. Then  $X_{(p)}$  admits a handle decomposition*

$$X_{(p)} = \text{one } 0\text{-handle} \cup (h_2 + 1) \text{ 2-handles} \cup h_3 \text{ 3-handles} \cup \text{one } 4\text{-handle}.$$

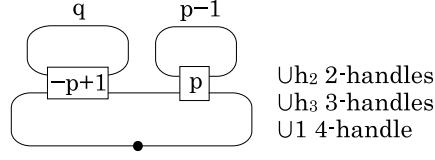
*In particular  $X_{(p)}$  admits a handle decomposition without 1-handles.*

FIGURE 6. Handle decomposition of  $X$ 

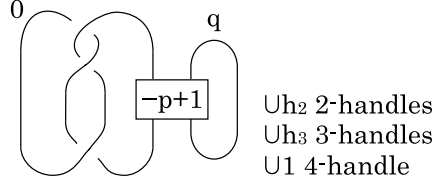
*Proof.* Draw a Kirby diagram of  $X_{(p)}$ , following the procedure introduced in Gompf-Stipsicz [4, Section 8.5] (see also [4, page 516 Solution of Exercise 8.5.1.(a)]). Then we have a handle decomposition of  $X_{(p)}$  as in Figure 7. We easily get a meridian of the unique dotted circle by a handle slide. Thus we can cancel the 1-handle/2-handle pair.  $\square$

**Corollary 3.5.** *Suppose that a simply connected closed smooth 4-manifold  $X$  has a handle decomposition as in Figure 8. Here  $q$  is an arbitrary integer.  $h_2$  and  $h_3$  are arbitrary non-negative integers. Let  $X_p$  be the smooth 4-manifold obtained from  $X$  by performing a logarithmic transformation of multiplicity  $p$  in the obvious cusp neighborhood in Figure 8. Then  $X_p$  admits a handle decomposition*

$$X_p = \text{one } 0\text{-handle} \cup (h_2 + 2) \text{ 2-handles} \cup h_3 \text{ 3-handles} \cup \text{one } 4\text{-handle}.$$


 FIGURE 7. Handle decomposition of  $X_{(p)}$ 

In particular  $X_p$  admits a handle decomposition without 1-handles.


 FIGURE 8. Handle decomposition of  $X$ 

*Proof.* Construct  $C_p$  from Figure 8, following the procedure given by Fintushel-Stern [2, Example 1] (and Gompf-Stipsicz [4, Section 8.5]). Then we have an embedding of  $C_p$  into  $X \# (p-1) \overline{\mathbf{CP}^2}$  such that the rational blow-down of  $X \# (p-1) \overline{\mathbf{CP}^2}$  along  $C_p$  is diffeomorphic to  $X_p$ . This embedding of  $C_p$  clearly satisfies the assumption of Lemma 3.4. Therefore we get the required handle decomposition of  $X_p$ .  $\square$

**Remark 3.6.** One can prove Corollary 3.5 without using rational blow-downs. Follow the procedure given by Gompf [3, Section 4].

Propositions 3.1, 3.2 and 3.3 together with Corollary 3.5 clearly give the following main theorem:

**Theorem 3.7.** *For  $n \geq 1$  and  $(p, q) = (2, 3), (2, 5), (3, 4), (4, 5)$ , the elliptic surface  $E(n)_{p,q}$  has a handle decomposition*

$$\text{one 0-handle} \cup 12n \text{ 2-handles} \cup \text{two 3-handles} \cup \text{one 4-handle}.$$

#### 4. FURTHER REMARKS

We finish this paper by making some remarks.

**Remark 4.1.** A key of our proof of the main theorem is to eliminate extra twists of a 2-handle of  $E(n)_p$  so that we can apply Corollary 3.5. To carry out the key, we used many vertical  $-1$  framed 2-handles of  $E(n)_p$  in Figure 9 or 35. Perhaps, we may obtain more examples of elliptic surfaces without 1-handles by additionally using horizontal 2-handles of  $E(n)_p$  in Figure 9 or 35.

**Remark 4.2.** In [7] and [8], we constructed a smooth 4-manifold  $E'_3$  which is homeomorphic to  $E(1)_{2,3}$ . The 4-manifold  $E'_3$  has the same Seiberg-Witten invariant as  $E(1)_{2,3}$  and has a handle decomposition without 1- and 3-handles.  $E'_3$  is constructed from  $\mathbf{CP}^2 \# 13 \overline{\mathbf{CP}^2}$  by rationally blowing down  $C_5$ . However, it is

not known whether or not  $E(1)_{2,3}$  can be obtained from  $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}^2}$  by rationally blowing down  $C_5$ . We do not know whether or not manifolds in [8] are diffeomorphic to  $E(1)_{2,q}$  ( $q = 3, 5$ ).

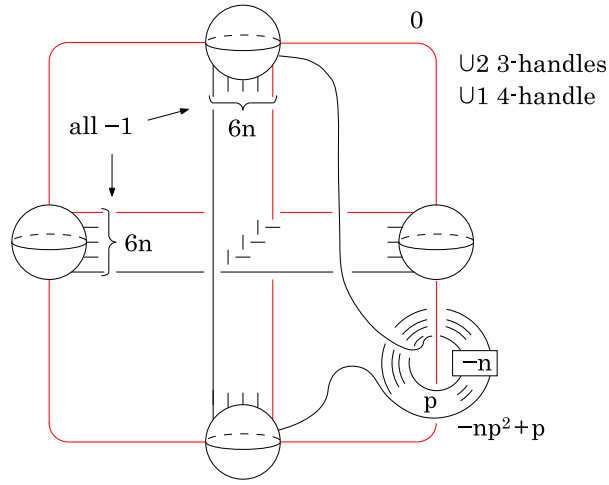
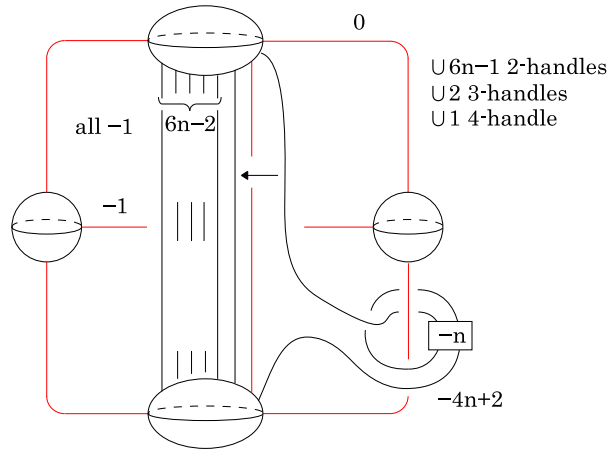
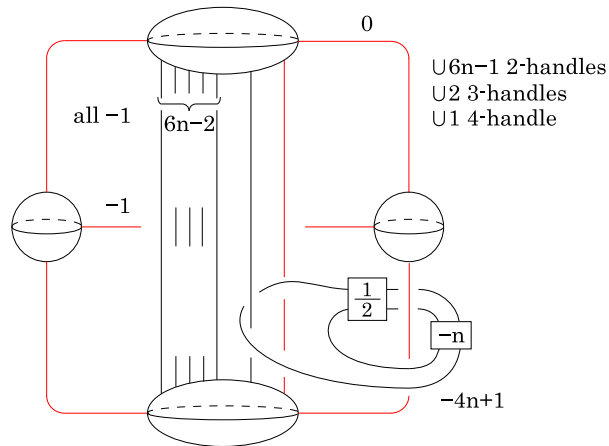
**Remark 4.3.** It seems more interesting to investigate handle decompositions of exotic 4-manifolds with small Euler characteristics, because there exist no exotic  $S^4$  and no exotic  $\mathbf{CP}^2$  which admit handle decompositions without 1- and 3-handles. In [9], we constructed exotic  $\mathbf{CP}^2 \# n\overline{\mathbf{CP}^2}$  ( $5 \leq n \leq 9$ ) which admit neither 1- nor 3-handles for  $7 \leq n \leq 9$ .

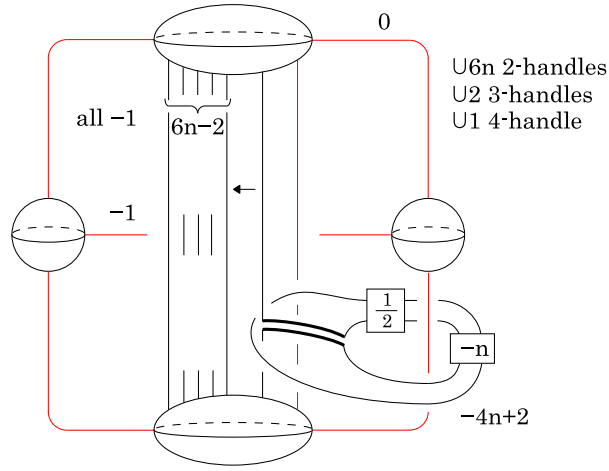
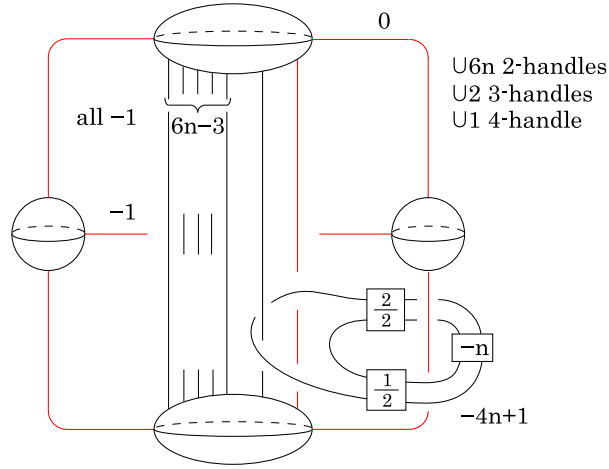
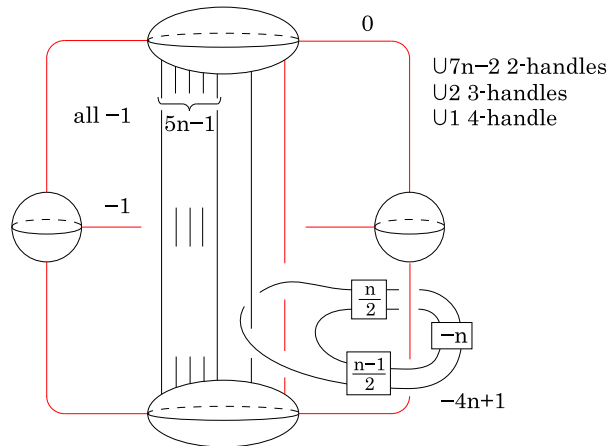
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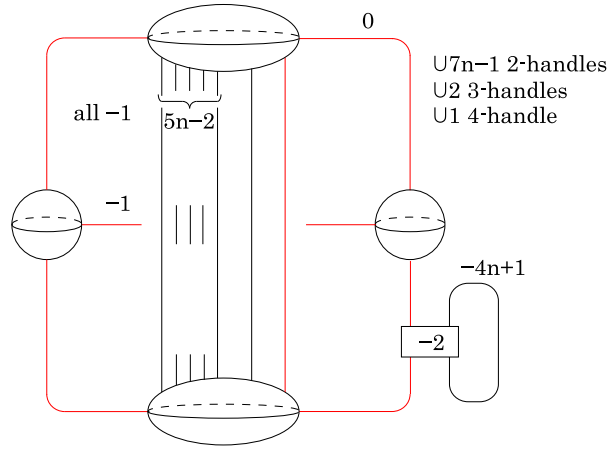
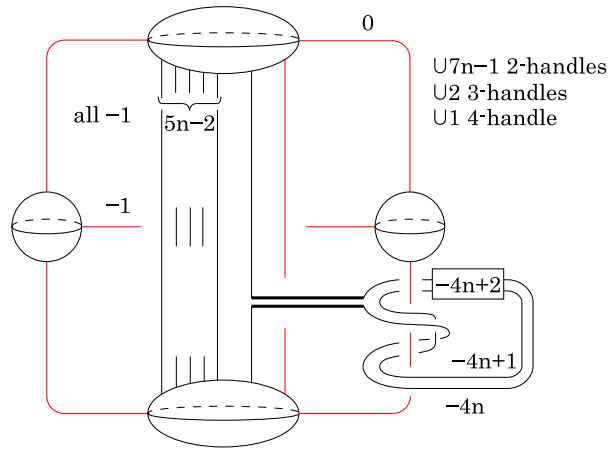
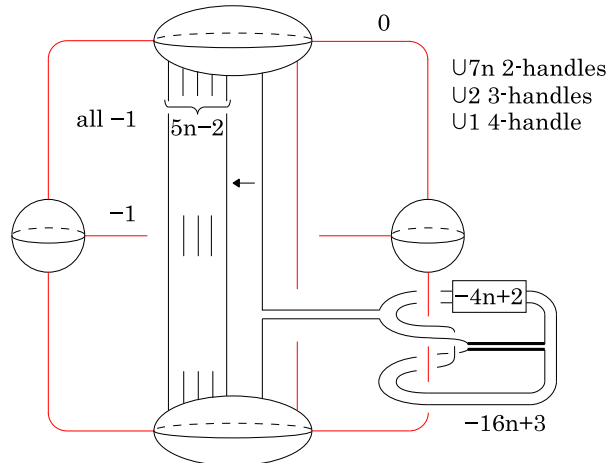
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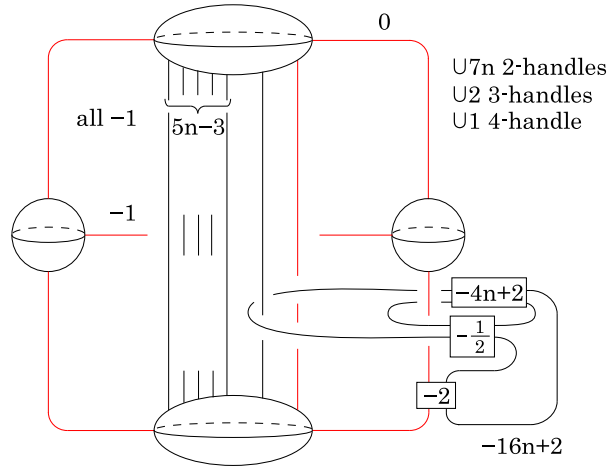
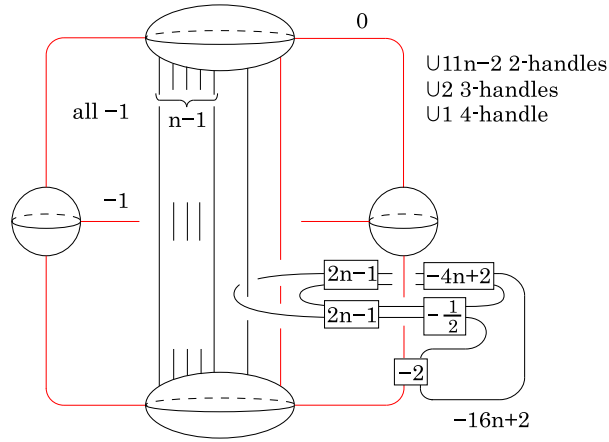
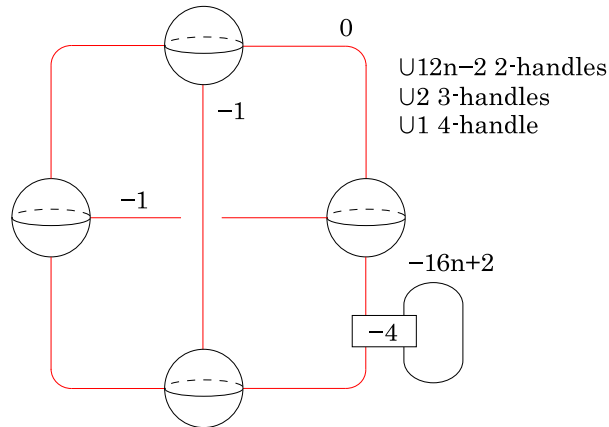
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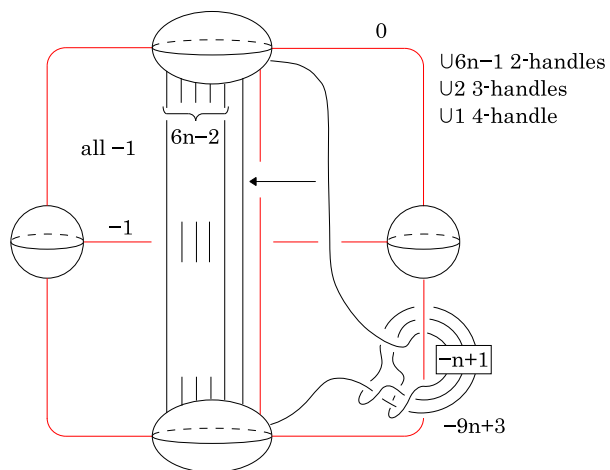
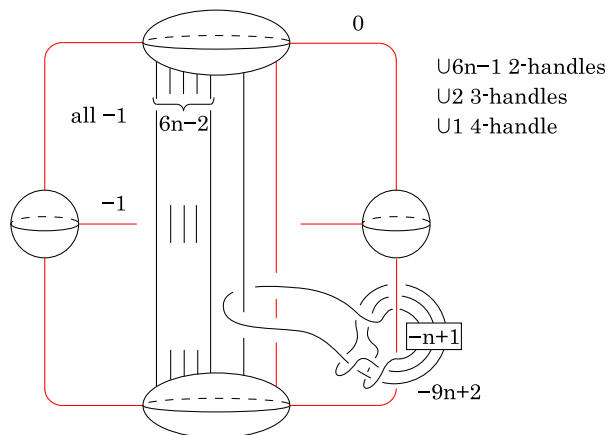
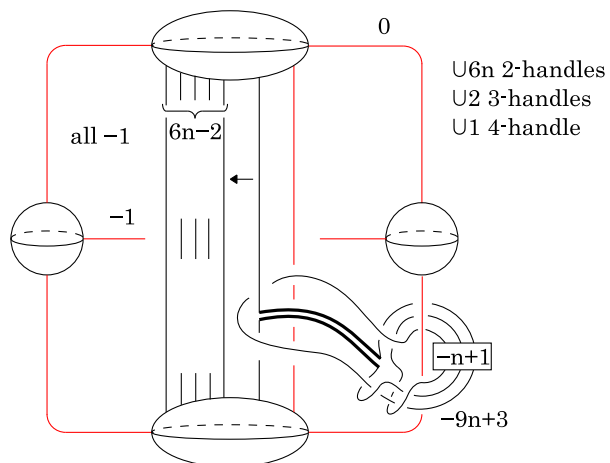

 FIGURE 9.  $E(n)_p$ 

 FIGURE 10.  $E(n)_2$ 

 FIGURE 11.  $E(n)_2$

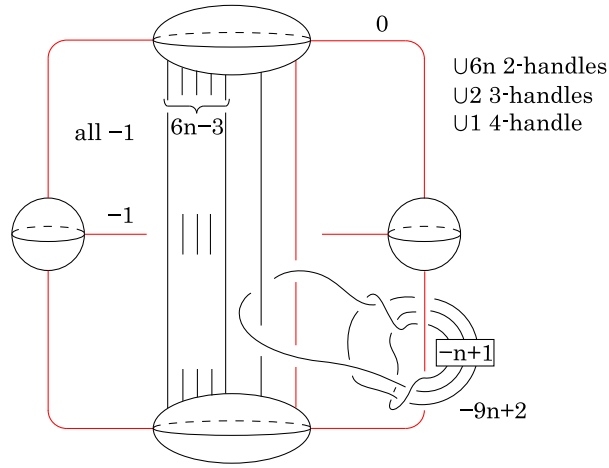
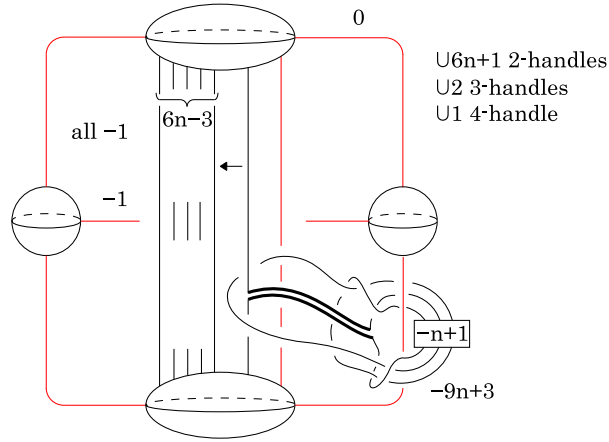
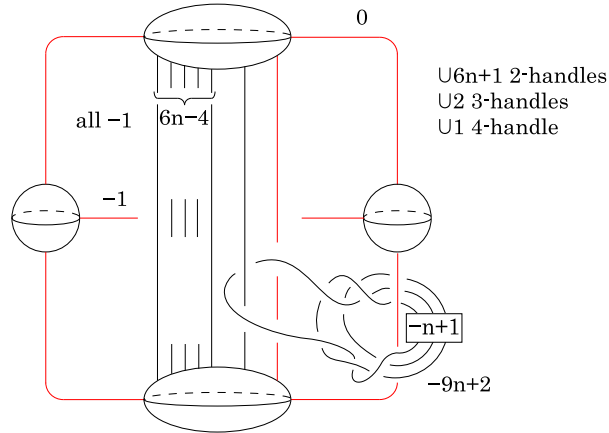
FIGURE 12.  $E(n)_2$ FIGURE 13.  $E(n)_2$ FIGURE 14.  $E(n)_2$

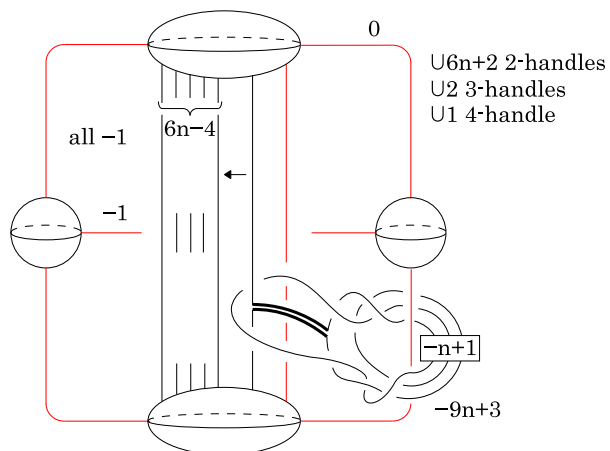
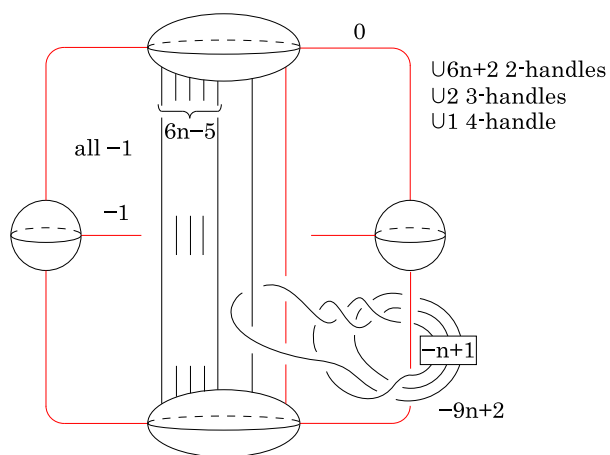
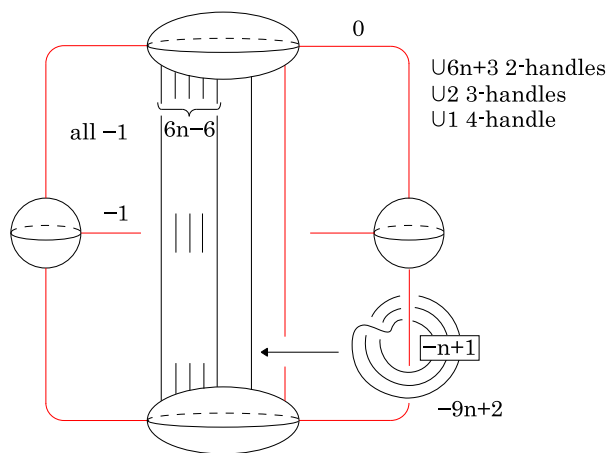


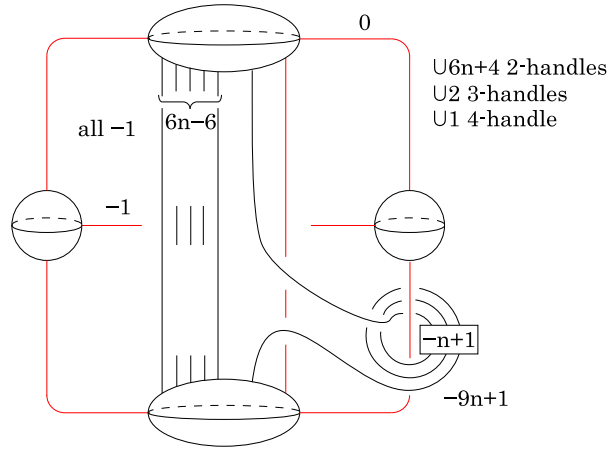
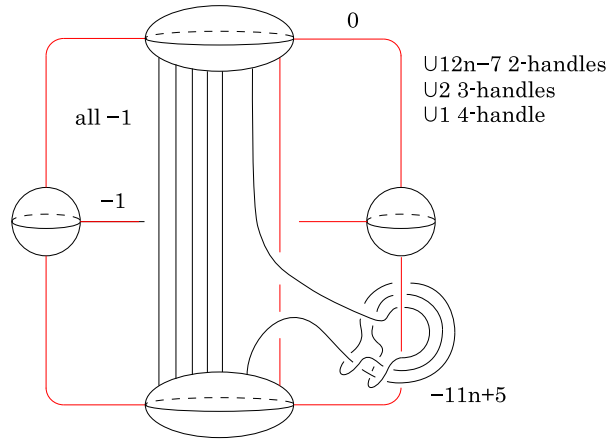
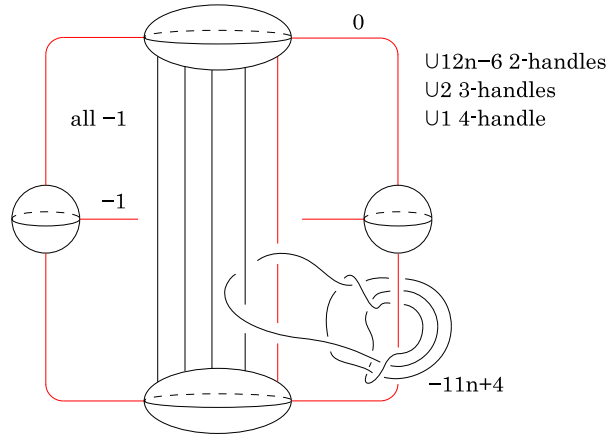

 FIGURE 15.  $E(n)_2$ 

 FIGURE 16.  $E(n)_2$ 

 FIGURE 17.  $E(n)_2$

FIGURE 18.  $E(n)_2$ FIGURE 19.  $E(n)_2$ FIGURE 20.  $E(n)_2$


 FIGURE 21.  $E(n)_3$ 

 FIGURE 22.  $E(n)_3$ 

 FIGURE 23.  $E(n)_3$

FIGURE 24.  $E(n)_3$ FIGURE 25.  $E(n)_3$ FIGURE 26.  $E(n)_3$


 FIGURE 27.  $E(n)_3$ 

 FIGURE 28.  $E(n)_3$ 

 FIGURE 29.  $E(n)_3$

FIGURE 30.  $E(n)_3$ FIGURE 31.  $E(n)_3$ FIGURE 32.  $E(n)_3$

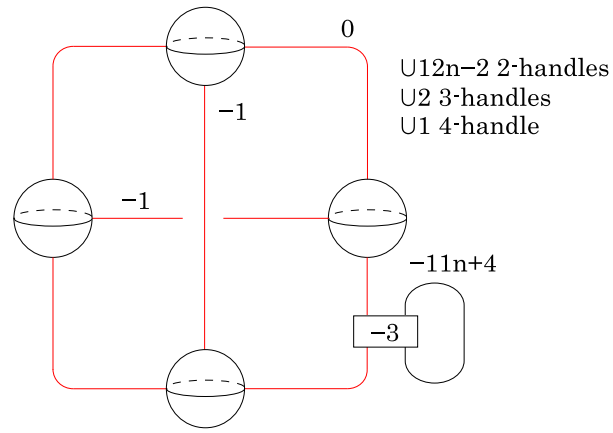
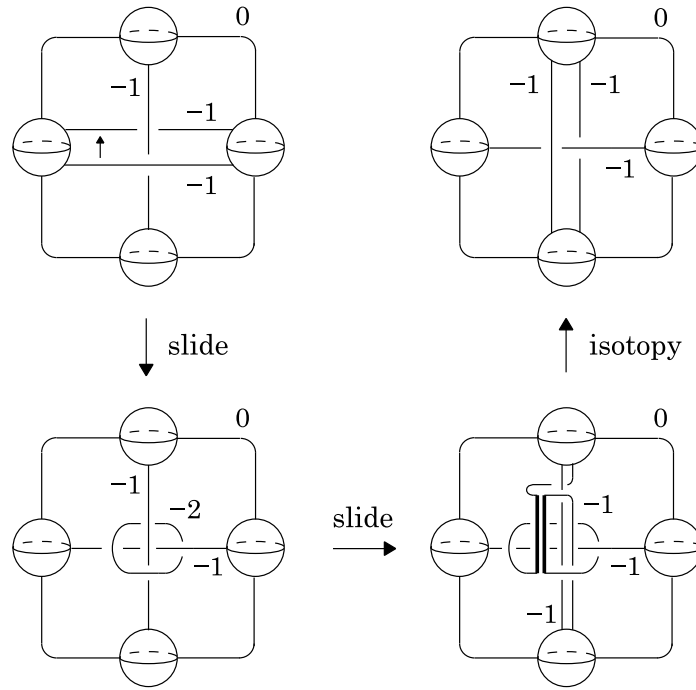
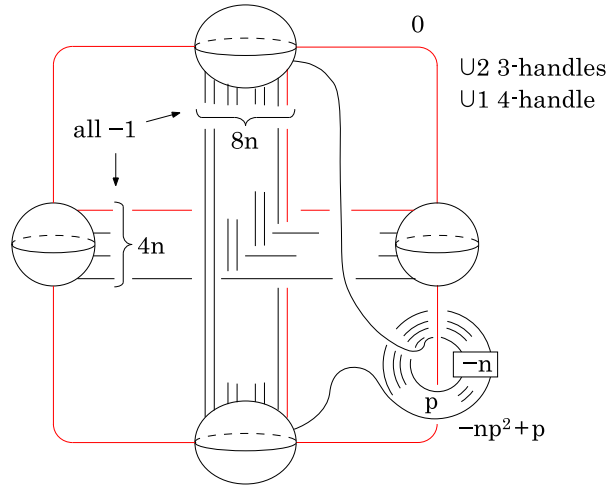
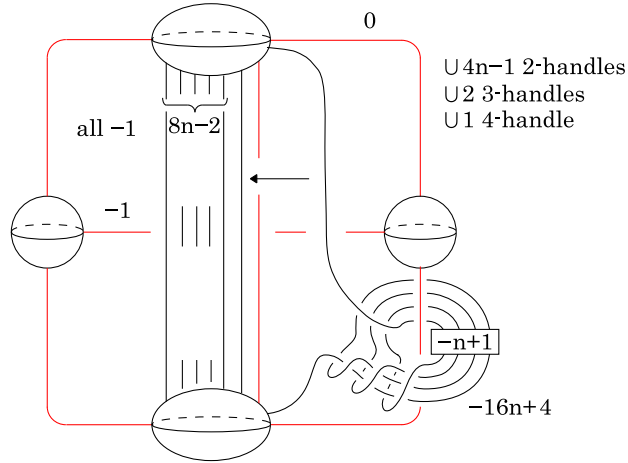
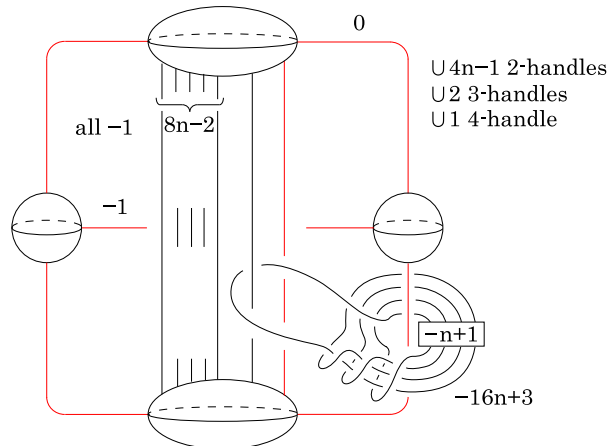
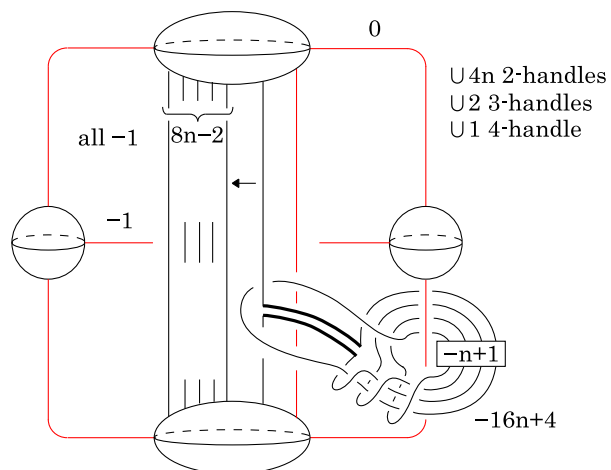
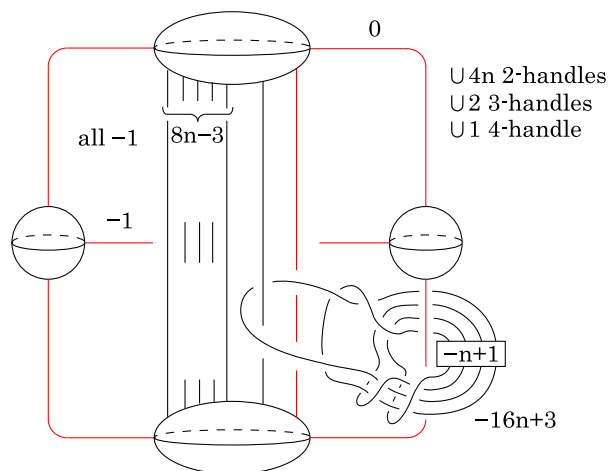
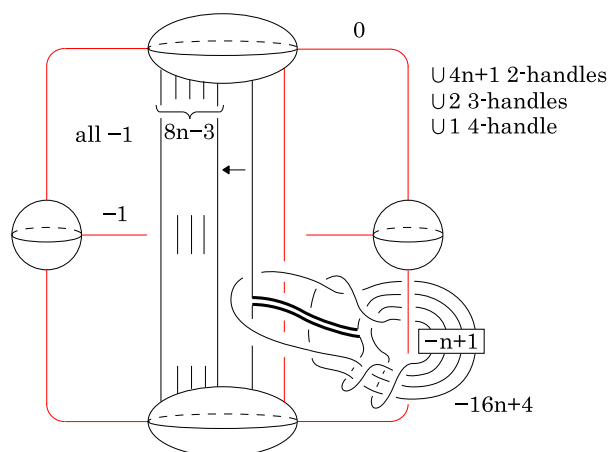
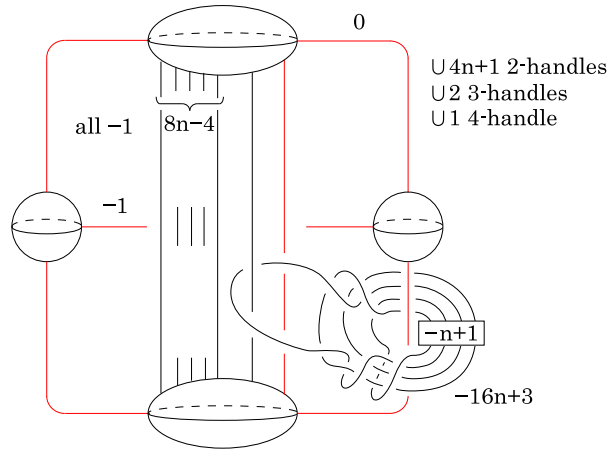
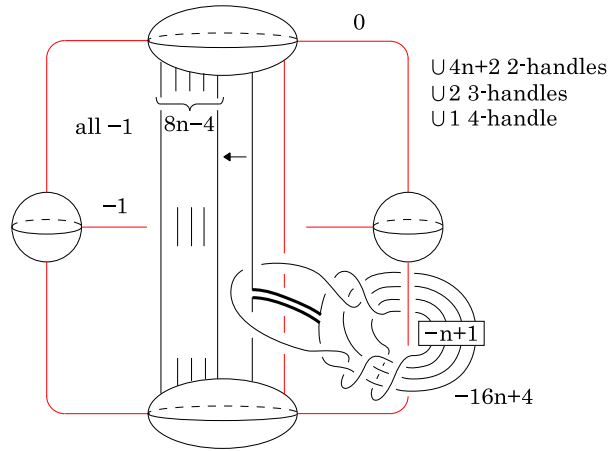
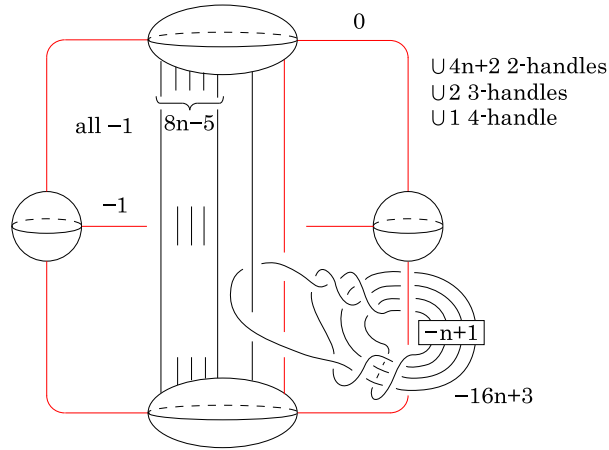

 FIGURE 33.  $E(n)_3$ 


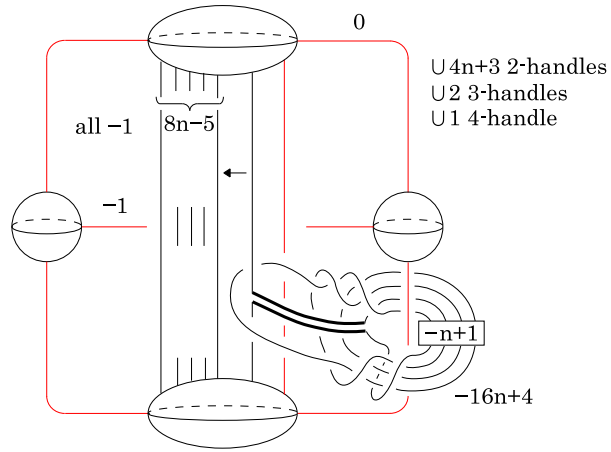
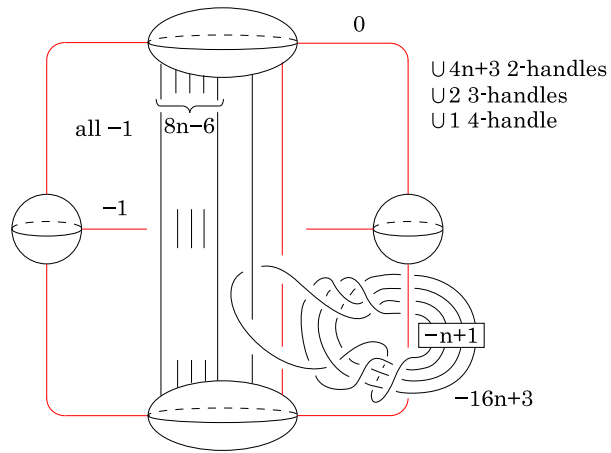
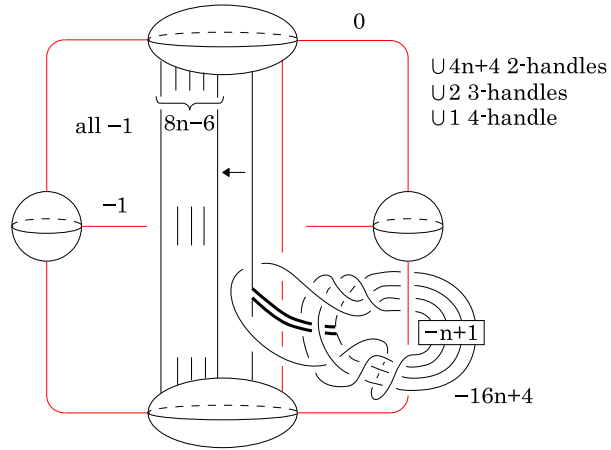
FIGURE 34. handle slides

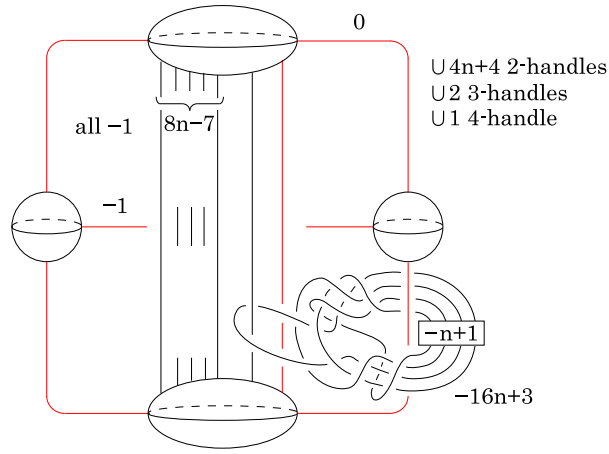
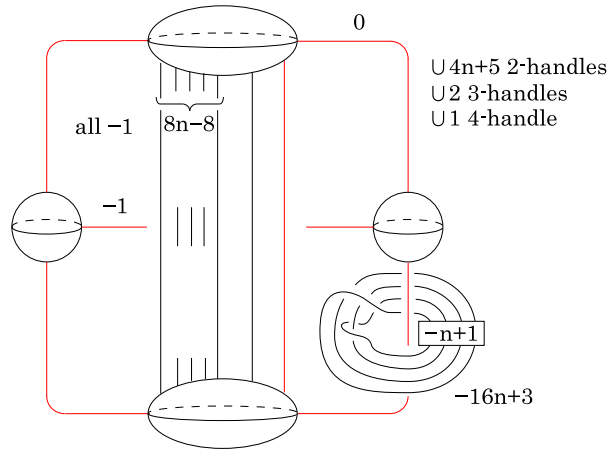
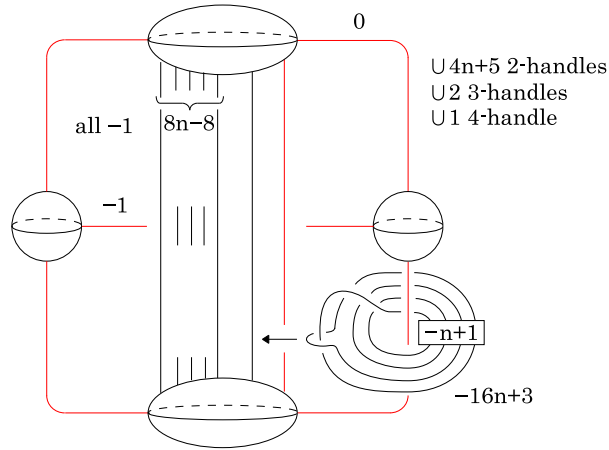
FIGURE 35.  $E(n)_p$ FIGURE 36.  $E(n)_4$ FIGURE 37.  $E(n)_4$

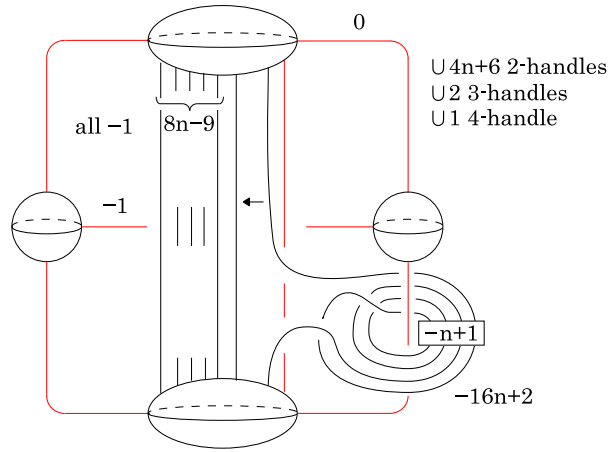
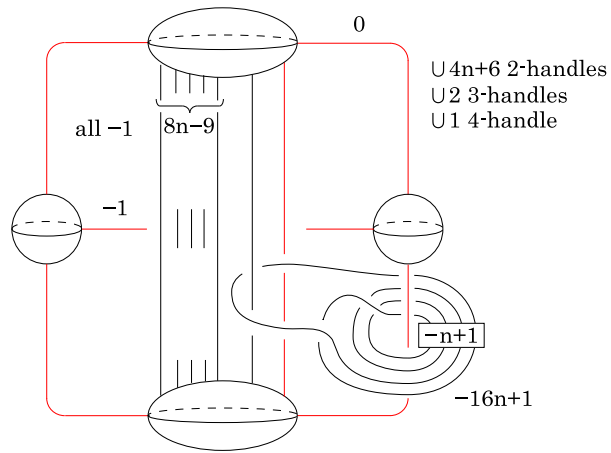
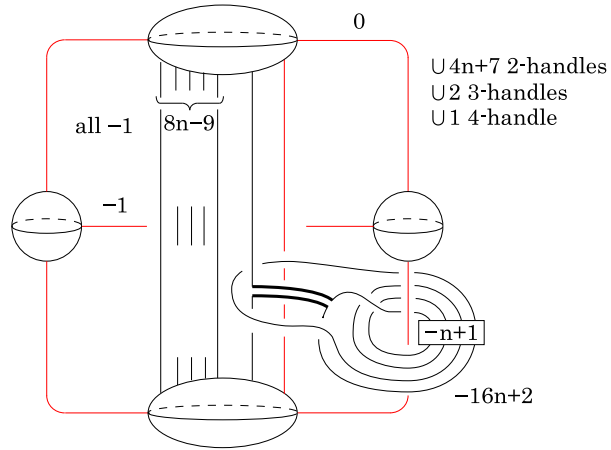


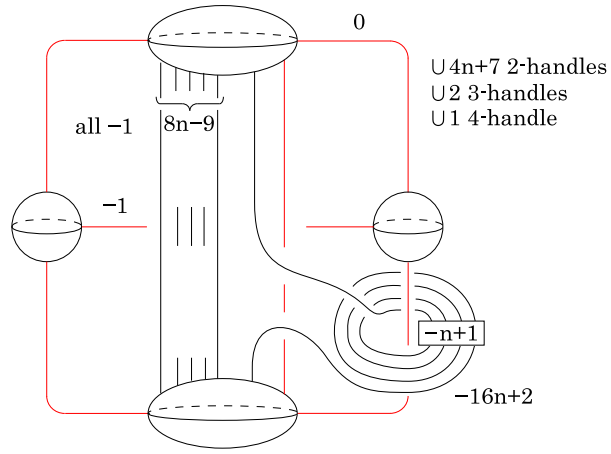
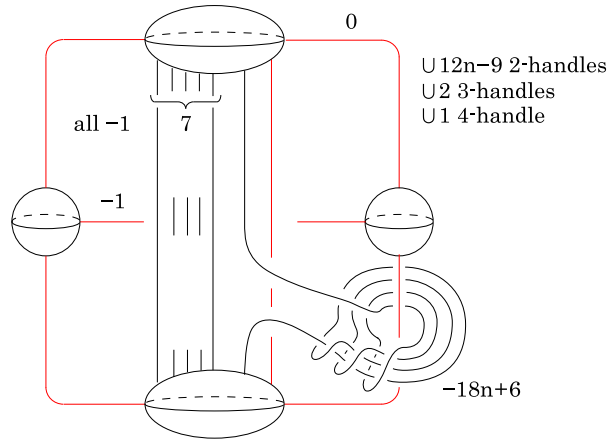
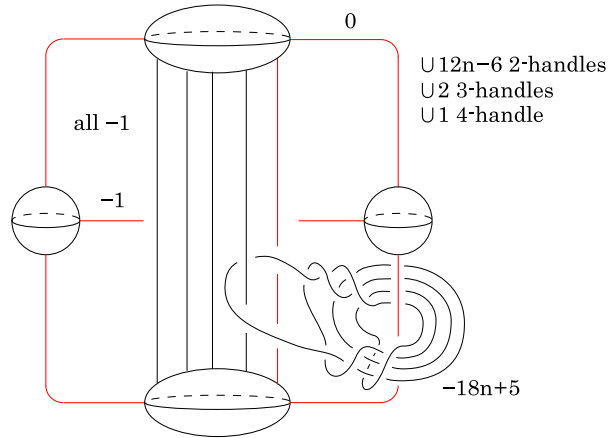

 FIGURE 38.  $E(n)_4$ 

 FIGURE 39.  $E(n)_4$ 

 FIGURE 40.  $E(n)_4$

FIGURE 41.  $E(n)_4$ FIGURE 42.  $E(n)_4$ FIGURE 43.  $E(n)_4$


 FIGURE 44.  $E(n)_4$ 

 FIGURE 45.  $E(n)_4$ 

 FIGURE 46.  $E(n)_4$

FIGURE 47.  $E(n)_4$ FIGURE 48.  $E(n)_4$ FIGURE 49.  $E(n)_4$


 FIGURE 50.  $E(n)_4$  ( $n \geq 2$ )

 FIGURE 51.  $E(n)_4$  ( $n \geq 2$ )

 FIGURE 52.  $E(n)_4$  ( $n \geq 2$ )

FIGURE 53.  $E(n)_4$  ( $n \geq 2$ )FIGURE 54.  $E(n)_4$ FIGURE 55.  $E(n)_4$

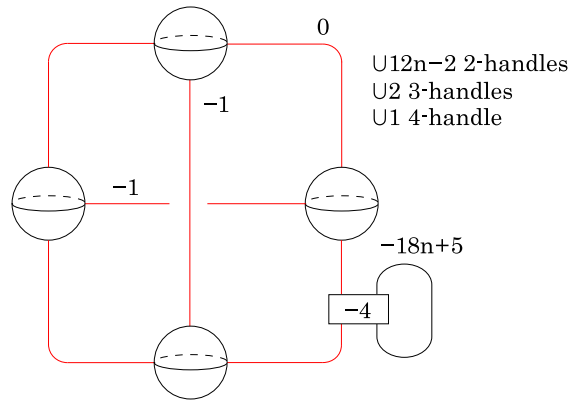


FIGURE 56.  $E(n)_4$